Math 2850 Laplace Transform Summary

Recall the definition of the Laplace Transform of a function $f(t)$:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

From class, we know the basic transforms:

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2} \qquad \mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \qquad \mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$$

Linearity: $\mathcal{L}\left\{c_1 f(t) \pm c_2 g(t)\right\} = c_1 \mathcal{L}\left\{f(t)\right\} \pm c_2 \mathcal{L}\left\{g(t)\right\} = c_1 F(s) \pm c_2 G(s)$

Forward Exponential Shift: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{e^{at} f(t)\right\} = \mathcal{L}\left\{f(t)\right\}_{s \to (s-a)} = F(s)_{s \to (s-a)} = F(s-a)$.

Transforms with Step Functions and Impulses: If $\mathcal{L}\{f(t)\} = F(s)$:

- $\mathcal{L}\{\mathcal{U}(t-a) f(t-a)\} = e^{-as} F(s)$. In particular, $\mathcal{L}\{\mathcal{U}(t-a)\} = \dfrac{e^{-as}}{s}$ and $\mathcal{L}\left\{\mathcal{U}(t)\right\} = \dfrac{1}{s}$
- $\mathcal{L}\{\delta(t-a)\} = e^{-as}$ where δ is the Dirac Delta 'function.' In particular, $\mathcal{L}\{\delta(t)\} = 1$
- $\mathcal{L}\{\mathcal{U}(t-a) f(t)\} = e^{-as} \mathcal{L}\{f(t+a)\}$

Transforms of Derivatives: If $\mathcal{L}\{y(t)\} = Y(s)$, then $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$ and $\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$

Derivatives of Transforms Theorem: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$.

Covolution: If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$.

Math 2850 Inverse Laplace Transform Summary

From class, we know the basic inverse transforms:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt) \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt) \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$$

Linearity: $\mathcal{L}^{-1}\left\{c_1F(s) \pm c_2G(s)\right\} = c_1\mathcal{L}^{-1}\left\{F(s)\right\} \pm c_2\mathcal{L}^{-1}\left\{G(s)\right\} = c_1f(t) \pm c_2g(t)$

Backwards Exponential Shift: $\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}\mathcal{L}^{-1}\left\{F(s)\right\} = e^{at}f(t)$ or $\mathcal{L}^{-1}\left\{F(s)\right\} = e^{at}\mathcal{L}^{-1}\left\{F(s+a)\right\}$

Backwards Unit Step Shift: $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = \mathcal{U}(t-a)\mathcal{L}^{-1}\left\{F(s)\right\}_{t\to(t-a)} = \mathcal{U}(t-a)f(t-a)$

Covolution: If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$: $\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = \mathcal{L}^{-1}\left\{F(s)\right\} * \mathcal{L}^{-1}\left\{G(s)\right\} = (f*g)(t)$